

# ANALYSIS OF THE ELECTRON PINCH DURING A BUNCH PASSAGE

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## Abstract

We present an analytical calculation of the radial electron density along a proton bunch passing through an electron cloud, considering various longitudinal distributions, a linear transverse force, and a round beam. From the electron density, we then infer the incoherent tune spread inside the bunch. The analytical results are compared with a computer simulation, by which we can also study the effect of a non-uniform transverse beam distribution.

## INTRODUCTION

During the passage of a proton (or positron) bunch through an electron cloud, the electrons are accumulated around the beam center. This pinch effect produces tune shift and tune spread in the bunch that could cause a slow incoherent emittance growth over successive turns. We compute the electron-cloud density evolution during a bunch passage and from this we infer the tune shift induced on the beam particles.

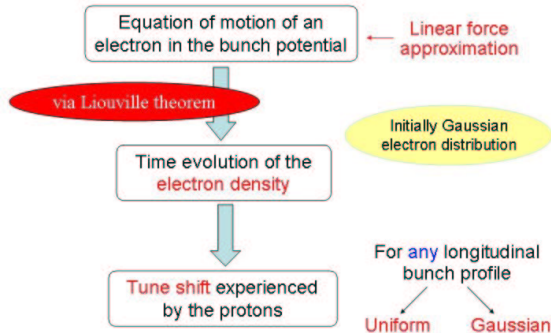


Figure 1: Flow diagram of the analytical calculation.

Figure 1 shows a schematic of our analytical procedure. Specifically, we consider a cylindrically symmetric model of a bunch passing through an electron cloud. We first solve the equations of motion of a single electron in the bunch potential under the simplifying approximation of a linear transverse force, for several longitudinal bunch profiles. Next, assuming an initially Gaussian electron distribution of finite temperature in transverse phase space, we compute the evolution of the electron density during the bunch passage, using Liouville's theorem. Finally, from the electron distribution so obtained we calculate the tune shift experienced by individual protons as a function of their transverse and longitudinal position. An explicit analytical solution is

derived for an arbitrary longitudinal profile, under the approximation of a linear transverse force. Approximations for low electron temperature are discussed.

Via computer simulation we extend the analysis to a non-linear transverse force for a Gaussian transverse beam profile. From the simulation result, we estimate the incoherent tune spread in the LHC.

## ELECTRON DENSITY EVOLUTION IN THE APPROXIMATION OF A LINEAR FORCE

We start from the electron distribution in the four-dimensional transverse phase space. In the linear force approximation, the horizontal and vertical planes are uncoupled. We thus factorize the electron density distribution as follows:

$$\rho(x, \dot{x}, y, \dot{y}, t) = \rho_x(x, \dot{x}, t) \rho_y(y, \dot{y}, t). \quad (1)$$

The spatial density also factorizes, as

$$n_e(r, t) \equiv n_e(x, y, t) = n_x(x, t) n_y(y, t), \quad (2)$$

where the projected spatial densities are obtained by integrating the projected phase-space densities over the electron velocities:

$$n_x(x, t) = \int d\dot{x} \rho_x(x, \dot{x}, t). \quad (3)$$

By our symmetry assumption  $n_e$  depends on  $x$  and  $y$  only in terms of the radius  $r \equiv \sqrt{x^2 + y^2}$ . From Liouville's theorem, we know that the electron density in the phase space is locally preserved. Hence, with the hypothesis of an initially Gaussian distribution for the electrons in their transverse phase space, we can write for the horizontal distribution

$$\rho_x(x, \dot{x}, t) = \rho_x(x_0, \dot{x}_0, 0) = \frac{\sqrt{\lambda_e}}{2\pi\sigma_0\dot{\sigma}_0} e^{-\frac{x_0^2}{2\sigma_0^2}} e^{-\frac{\dot{x}_0^2}{2\dot{\sigma}_0^2}}, \quad (4)$$

and an analogous expression applies to the vertical plane. Here, the parameters  $\sigma_0$  and  $\dot{\sigma}_0$  denote the horizontal rms size of the initial electron distribution and its horizontal rms velocity, respectively. For the circularly symmetric problem that we consider here, the vertical density has the same form with identical rms size and velocity. We will later obtain some approximate compact expressions for the special case that the initial velocities of the electrons are small compared with the (correlated) velocities acquired in the beam potential, i.e.  $\dot{\sigma}_0 \ll \omega_e \sigma_0$ .

If we are able to solve and invert the equation of motion of a single electron in the bunch potential, we can express

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$(x_0, \dot{x}_0)$  as a function of  $(x, \dot{x}, t)$  and insert these resulting expressions into the right-hand side of (4) in order to get the electron density at the time  $t$ <sup>1</sup>.

### Equation of motion of an electron in the bunch potential

We assume that the transverse bunch distribution is Gaussian and cylindrically symmetric, characterized by the rms transverse beam size  $\sigma_r$  (namely  $\sigma_r = \sigma_x = \sigma_y$ ):

$$\tilde{\rho}_b(r, z) = \frac{e \tilde{\lambda}_b(z)}{2\pi\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}}, \quad (5)$$

where  $\tilde{\lambda}_b(z)$  is the beam-charge line density, which depends on the longitudinal distance from the bunch center,  $z$ . We can then find the radial electrical field of the beam experienced by the electrons applying Gauss' theorem for a cylinder of radius  $r$ ,

$$2\pi r \tilde{E}_b(r, z) = \frac{1}{\epsilon_0} \int_0^r \tilde{\rho}_b(r', z) 2\pi r' dr', \quad (6)$$

which gives

$$\tilde{E}_b(r, z) = \frac{e \tilde{\lambda}_b(z)}{2\pi\epsilon_0 r} \left[ 1 - e^{-\frac{r^2}{2\sigma_r^2}} \right]. \quad (7)$$

For comparison with simulations, presented later, we express the field as a function of time

$$t = \frac{1}{c}(n\sigma_z - z), \quad (8)$$

where  $t = 0$  refers to the instant when the bunch enters into the e-cloud. We will use  $n = 3$ . This yields

$$E_b(r, t) = \frac{e \lambda_b(t)}{2\pi\epsilon_0 r} \left[ 1 - e^{-\frac{r^2}{2\sigma_r^2}} \right] \quad (9)$$

In radial coordinates, the equation of motion of an electron in the bunch potential is

$$\begin{aligned} m_e \frac{d^2 r}{dt^2} &= -e E_b(r, t) + \frac{l^2}{m_e r^3} \\ &= -\frac{e^2 \lambda_b(t)}{2\pi\epsilon_0 r} \left[ 1 - e^{-\frac{r^2}{2\sigma_r^2}} \right] + \frac{l^2}{m_e r^3}, \end{aligned} \quad (10)$$

where  $l = mr^2\dot{\phi}$  denotes the angular momentum, which is a constant of motion.

Expressed in cartesian coordinates the angular momentum term disappears:

$$\begin{aligned} \ddot{x}(t) &= -\frac{2r_e c^2 \lambda_b(t) x}{r^2} \left[ 1 - e^{-\frac{r^2}{2\sigma_r^2}} \right], \\ \ddot{y}(t) &= -\frac{2r_e c^2 \lambda_b(t) y}{r^2} \left[ 1 - e^{-\frac{r^2}{2\sigma_r^2}} \right], \end{aligned} \quad (11)$$

We introduced the classical electron radius

$$r_e \equiv \frac{e^2}{4\pi\epsilon_0 m_e c^2}. \quad (12)$$

<sup>1</sup>A similar method was used in [1] to compute the beam density evolution under the influence of nonlinear field errors.

### Approximation of Linear Force

Under the linear approximation ( $r \ll \sigma_r$ ) the motion in the two transverse planes is decoupled and the equation in the horizontal plane is:

$$\ddot{x} + \omega_e^2(t)x = 0 \quad (13)$$

$$\omega_e^2(t) = \frac{\lambda_b(t)r_e c^2}{\sigma_r^2}. \quad (14)$$

A similar expressions holds for the vertical plane. With the linear approximation, it is possible to solve the equation of motion (13) and invert the solution, getting  $(x_0, \dot{x}_0)$  as a function of  $(x, \dot{x})$  in the form:

$$\begin{aligned} x_0 &= a(t)x + b(t)\dot{x} \\ \dot{x}_0 &= c(t)x + d(t)\dot{x}, \end{aligned} \quad (15)$$

where the coefficients  $a(t), \dots, d(t)$  depend on the longitudinal distribution, and for a conservative system ( $ad - bc = 1$ ). The electron distribution in phase space is computed by inserting (15) into (4) and the spatial electron density evolution is obtained by integrating over the velocities:

$$n_x(x, t) = \int_{-\infty}^{+\infty} d\dot{x} \rho_x(x, \dot{x}, t) \quad (16)$$

### Tune Shift

From the electron density we can compute the tune shift induced on the beam (over one turn around the ring).

The electrical field produced by the electrons can again be obtained by Gauss' theorem:

$$2\pi r E_e(r, t) = \frac{-|e|}{\epsilon_0} \int_0^r n_e(r', t) 2\pi r' dr', \quad (17)$$

As we are interested in the field experienced by a proton at position  $(r, z)$  in the bunch, we need to return to the coordinate  $z$ :  $z = n\sigma_z - ct$ . Then, we get:

$$\tilde{E}_e(r, z) = \frac{-|e|\tilde{\lambda}(r, z)}{2\pi\epsilon_0 r} \quad (18)$$

with

$$\tilde{\lambda}(r, z) = 2\pi \int_0^r \tilde{n}_e(r', z) r' dr' \quad (19)$$

where

$$\tilde{n}_e(r', z) = n_e(r', t = \frac{1}{c}(n\sigma_z - z))$$

The tune shift in the horizontal plane is

$$\Delta Q_x = \frac{1}{4\pi} \oint_C ds \beta(s) \Delta k_x, \quad (20)$$

where

$$\Delta k_x = -\frac{e}{\gamma m_p c^2} \frac{\partial \tilde{E}_{e,x}}{\partial x}. \quad (21)$$

As we have

$$\frac{\partial \tilde{E}_{e,x}}{\partial x} = \frac{\partial \tilde{E}_{e,r}}{\partial r} \bigg|_{y=0}, \quad (22)$$

we obtain

$$\begin{aligned} \Delta k_x &= -\frac{e}{\gamma m_p c^2} \frac{\partial \tilde{E}_{e,r}}{\partial r} \\ &= \frac{4\pi r_p}{\gamma} \left[ \tilde{n}_e(r, z) - \frac{1}{2\pi r^2} \tilde{\lambda}(r, z) \right] \\ &= \frac{4\pi r_p}{\gamma} \left[ \tilde{n}_e(r, z) - \frac{1}{r^2} \int \tilde{n}_e(r', z) r' dr' \right] \end{aligned} \quad (23)$$

and the tune shift becomes

$$\begin{aligned} \Delta Q_x(r, z) &= \\ &= \oint_C ds \beta(s) \frac{r_p}{\gamma} \left[ \tilde{n}_e(r, z) - \frac{1}{r^2} \int \tilde{n}_e(r', z) r' dr' \right]. \end{aligned} \quad (24)$$

In principle, the proton beam size depends on the beta functions and, thus, also the electron density  $\tilde{n}_e(r, z)$  depends on the position around the ring  $s$ . In the following we will use the smooth focusing approximation,

$$\beta(s) = \text{const.} = \bar{\beta}, \quad (25)$$

and we also assume a constant electron-cloud density, so that the integrand becomes independent of  $s$ , and the integral over  $s$  amounts to a multiplication by the circumference  $C$ .

We now derive the tune shift for two specific longitudinal bunch profiles and for the general case.

### Longitudinal Uniform Bunch Profile, Linear Force

In the case

$$\lambda_b(t) = \bar{\lambda}_b = \text{const.}$$

the equation of motion reduces to the harmonic oscillator:

$$\begin{aligned} \ddot{x} + \omega_e^2 x &= 0 \\ \omega_e^2 &= \frac{\bar{\lambda}_b r_e c^2}{\sigma_r^2}, \end{aligned}$$

with the solution

$$\begin{aligned} x(t) &= x_0 \cos(\omega_e t) + \dot{x}_0 \frac{1}{\omega_e} \sin(\omega_e t) \\ \dot{x}(t) &= -x_0 \omega_e \sin(\omega_e t) + \dot{x}_0 \cos(\omega_e t). \end{aligned} \quad (26)$$

Inverting (26) we obtain  $(x_0, \dot{x}_0)$  as a function of  $(x, \dot{x})$ :

$$\begin{aligned} x_0 &= x \cos(\omega_e t) - \dot{x} \frac{1}{\omega_e} \sin(\omega_e t) = C x - \frac{S}{\omega_e} \dot{x} \\ \dot{x}_0 &= x \omega_e \sin(\omega_e t) + \dot{x} \cos(\omega_e t) = \omega_e S x + C \dot{x} \end{aligned}$$

with

$$C = \cos(\omega_e t), \quad (27)$$

$$S = \sin(\omega_e t), \quad (28)$$

which we now insert into Eq. (4).

After some algebra, we obtain the electron distribution function at the time  $t$  along the bunch. It can be written as

$$\rho_x(x, \dot{x}, t) = A e^{-c(x,t)} e^{-a(t)(\dot{x}-b(x,t))^2} \quad (29)$$

where:

$$\begin{aligned} A &= \frac{\sqrt{\lambda_e}}{2\pi\sigma_0\dot{\sigma}_0} \\ c(x, t) &= \frac{\omega_e^2 x^2}{2(\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2)} \\ a(t) &= \frac{(\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2)}{2\omega_e^2 \sigma_0^2 \dot{\sigma}_0^2} \\ b(x, t) &= \frac{\omega_e x (\dot{\sigma}_0^2 - \omega_e \sigma_0^2) SC}{(\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2)} \end{aligned}$$

The density is obtained by integrating:

$$\begin{aligned} n_x(x, t) &= \int_{-\infty}^{+\infty} d\dot{x} \rho_x(x, \dot{x}, t) = \\ &= \frac{\sqrt{\pi\lambda_e}}{2\pi\sigma_0\dot{\sigma}_0} \frac{e^{-c(x,t)}}{\sqrt{a(t)}} \\ &= \frac{\sqrt{\lambda_e}\omega_e}{\sqrt{2\pi}} \frac{e^{-c(x,t)}}{\sqrt{\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2}} \end{aligned} \quad (30)$$

In the low-temperature or strong-beam limit,  $\dot{\sigma}_0 \ll \omega_e \sigma_0$ , and the above expression reduces to:

$$n_x(x, t) \approx \frac{\sqrt{\lambda_e}}{\sqrt{2\pi}\sigma_0} \frac{e^{-\frac{\omega_e^2 x^2}{2(\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2)}}}{|C|} \quad (31)$$

The total 2-dimensional density is just the product of  $n_x(x, t)$  and  $n_y(y, t)$ , namely

$$\begin{aligned} n_e(x, y, t) &= n_x(x, t) n_y(y, t) \\ &= \frac{\lambda_e \omega_e^2}{2\pi} \frac{e^{-\frac{\omega_e^2 r^2}{2(\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2)}}}{\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2} \\ &\approx \frac{\lambda_e}{2\pi\sigma_0^2} \frac{e^{-\frac{\omega_e^2 r^2}{2(\dot{\sigma}_0^2 S^2 + \omega_e^2 \sigma_0^2 C^2)}}}{C^2}, \end{aligned} \quad (32)$$

where the last expression refers to the low-temperature limit. The right-hand side depends only on  $r$ , as a result of the circular geometry.

Inserting

$$\tilde{n}_e(r, z) = n_e \left( r, t = \frac{1}{c}(n\sigma_z - z) \right) \quad (33)$$

into (23), and keeping only the lowest-order terms in  $(\omega_e r)$ , we get

$$\Delta k_x \approx \frac{\lambda_e r_p}{\gamma C^2 \sigma_0^2} \frac{1}{1 + \frac{S^2 \dot{\sigma}_0^2}{C^2 \omega_e^2 \sigma_0^2}} \left( 1 - \frac{\omega_e^2 r^2}{2(C^2 \omega_e^2 \sigma_0^2 + S^2 \dot{\sigma}_0^2)} \right) \quad (34)$$

where  $C(t)$  and  $S(t)$  are evaluated at  $t = (n\sigma_z - z)/c$ .

The tune shift (20) for a particle at position  $r$  and  $z$  in the bunch is

$$\Delta Q_x(r, z) \approx \frac{\bar{\beta} L \lambda_e r_p}{4\pi\gamma C^2 \sigma_0^2} \frac{1}{1 + \frac{S^2 \dot{\sigma}_0^2}{C^2 \omega_e^2 \sigma_0^2}} \times \left( 1 - \frac{\omega_e^2 r^2}{2(C^2 \omega_e^2 \sigma_0^2 + S^2 \dot{\sigma}_0^2)} \right) \quad (35)$$

where  $L$  is the circumference of the ring and the smooth focusing assumption has been invoked.

The tune shift depends on the longitudinal position with respect to the bunch center and it decreases parabolically with transverse distance  $r$ .

We note that for  $\dot{\sigma}_0 \ll \sigma_0 \omega_e$ , the tune shift becomes maximum at periodic intervals along the bunch, given by:

$$C \left( t = \frac{1}{c} (n\sigma_z - z) \right) = \cos \left( \omega_e z - \omega_e \frac{n\sigma_z}{c} \right) = 0$$

or

$$z = \frac{n\sigma_z}{c} + \frac{1}{\omega_e} (2k+1) \frac{\pi}{2}, \quad k = 0, 1, \dots$$

The maximum tune shift is:

$$\Delta \hat{Q}_x \approx \frac{\bar{\beta} L \lambda_e r_p \omega_e^2}{4\pi\gamma} \frac{1}{\dot{\sigma}_0^2}. \quad (36)$$

It depends inversely on the square of the initial electron velocity spread.

### Generalization to Arbitrary Distribution in $z$ , Linear Force

If the longitudinal distribution of the beam,  $\lambda_b(t)$ , is not a constant, the electron oscillation frequency varies with time  $t$  (14)

$$\omega_e^2(t) = \frac{\lambda_b(t) r_e c^2}{\sigma_r^2} \quad (37)$$

and the equation to be solved is (13):

$$\ddot{x} + \omega_e^2(t)x = 0 \quad (38)$$

If the change in oscillation frequency is adiabatic,

$$\left| \frac{3}{2} \frac{\dot{\omega}_e}{\omega_e} \right|, \quad \left| \frac{\ddot{\omega}_e}{\dot{\omega}_e} \right| \ll 2\omega_e, \quad (39)$$

we can apply the WKB approximation. Namely we look for a solution in the form

$$x(t) = A(t) e^{iS(t)}. \quad (40)$$

Substituting this ansatz into (13) and dropping small terms yields

$$A(t) \approx \frac{1}{\sqrt{\dot{S}(t)}} \quad (41)$$

and

$$\dot{S}(t) \approx \omega_e(t). \quad (42)$$

The general solution will be:

$$x(t) = \frac{c_1}{\sqrt{\omega_e(t)}} \cos S(t) + \frac{c_2}{\sqrt{\omega_e(t)}} \sin S(t), \quad (43)$$

where

$$S(t) = \int_0^t \omega_e(t) dt \quad (44)$$

and  $c_1$  and  $c_2$  are determined from the initial conditions  $x_0$  and  $\dot{x}_0$ .

In the general case of an arbitrary longitudinal bunch distribution  $\lambda_b(t)$  – but as before for a linear transverse force –, we can still invert the solution and determine  $(x_0, \dot{x}_0)$  as a function of  $(x, \dot{x})$ , as in Eq.(15), to insert the result into the expression of the electron distribution in phase space (4). The density  $n_e(r, t)$  is obtained by integrating over the velocities and it has the general form:

$$n_e(r, t) = \frac{\lambda_e}{2\pi D(t)} e^{-\frac{r^2}{2D(t)}} \quad (45)$$

with

$$D(t) = d(t)^2 \sigma_0^2 + b(t)^2 \dot{\sigma}_0^2 \quad (46)$$

which depends on the longitudinal profile of the bunch.

Again assuming the smooth focusing approximation (25), the tune shift in the horizontal or vertical plane has the general form:

$$\begin{aligned} \Delta Q_{x,y}(r, z) &= \frac{1}{4\pi} \bar{\beta} L \Delta k \\ &= \frac{\lambda_e r_p \bar{\beta} L}{2\pi\gamma r^2} \left( 1 - \frac{e^{-\frac{r^2}{2D}} (r^2 + D)}{D} \right). \end{aligned} \quad (47)$$

Expanding and keeping only the lowest-order terms in  $r^2/D$ , this simplifies to

$$\Delta Q_x(r, z) \approx \frac{\bar{\beta} L \lambda_e r_p}{4\pi\gamma D} \left( 1 - \frac{3}{4} \frac{r^2}{D} \right) \quad (48)$$

**Gaussian longitudinal shape** In the case of a bunch with a Gaussian longitudinal shape:

$$\tilde{\lambda}_b(z) = \frac{N_b}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}}; \quad z \in (-\infty, +\infty) \quad (49)$$

we have, as a function of time,

$$\lambda_b(t) = \frac{N_b}{\sqrt{2\pi}\sigma_z} e^{-\frac{(n\sigma_z - ct)^2}{2\sigma_z^2}} \quad (50)$$

$$\omega_e(t) = \Omega e^{-\frac{(n\sigma_z - ct)^2}{4\sigma_z^2}}$$

$$S(t) = \Omega \frac{\sigma_z \sqrt{\pi}}{c} \left\{ \text{Erf} \left( \frac{n}{2} \right) + \text{Erf} \left[ \frac{1}{2} \left( \frac{ct}{\sigma_z} - n \right) \right] \right\}$$

$$\Omega = \sqrt{\frac{r_e N_b c^2}{\sigma_r^2 \sigma_z \sqrt{2\pi}}}$$

Inverting the solution, we get from (15)

$$\begin{aligned}x_0 &= a(t)x + b(t)\dot{x} \\ \dot{x}_0 &= c(t)x + d(t)\dot{x},\end{aligned}$$

where the coefficient  $a, b, c, d$  are:

$$\begin{aligned}a(t) &= e^{\left[-\frac{\tilde{z}^2}{8} + \frac{n}{4}\tilde{z}\right]} \cos S(t) \\ &\quad + e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \frac{(\tilde{z} - n)}{4} \frac{c}{\Omega} \sin S(t) \\ b(t) &= -e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \frac{1}{\Omega} \sin S(t) \\ c(t) &= \left[ \frac{c(n - \tilde{z})}{4\sigma_z} e^{\left(\frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \right. \\ &\quad \left. - \frac{cn}{4\sigma_z} e^{\left(-\frac{\tilde{z}^2}{8} + \frac{n}{4}\tilde{z}\right)} \right] \cos S(t) + \\ &\quad + \left[ \Omega e^{-\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \right. \\ &\quad \left. + \frac{cn(n - \tilde{z})}{16\sigma_z^2} \frac{c}{\Omega} e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \right] \sin S(t) \\ d(t) &= e^{\left(\frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \cos S(t) \\ &\quad + e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z}\right)} \frac{n}{4\sigma_r} \frac{c}{\Omega} \sin S(t)\end{aligned}$$

with

$$\tilde{z} = \frac{ct}{\sigma_z}.$$

Using these expressions for  $d(t)$  and  $b(t)$ , the density and the tune shift are obtained from (48) with  $D(t)$  given by

$$\begin{aligned}D(t) &= d(t)^2 \sigma_0^2 + b(t)^2 \dot{\sigma}_0^2 = \\ &= \sigma_0^2 e^{\left(\frac{\tilde{z}^2}{4} - \frac{n}{2}\tilde{z}\right)} \cos^2 S(t) + \\ &\quad \sigma_0^2 e^{\left(\frac{n^2}{2} + \frac{\tilde{z}^2}{4} - \frac{n}{2}\tilde{z}\right)} \left(\frac{n}{4\sigma_r}\right)^2 \frac{c^2}{\Omega^2} \sin^2 S(t) + \\ &\quad - \sigma_0^2 e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{4} - \frac{n}{2}\tilde{z}\right)} \frac{n}{2\sigma_r} \frac{c}{\Omega} \sin S(t) \cos S(t) + \\ &\quad + \dot{\sigma}_0^2 \frac{1}{\Omega^2} e^{\left(\frac{n^2}{2} + \frac{\tilde{z}^2}{4} - \frac{n}{2}\tilde{z}\right)} \sin^2 S(t)\end{aligned}$$

The tune shift at the start of the bunch ( $\tilde{z} = 0$ ) is

$$\Delta Q_x(r, z) \approx \frac{\tilde{\beta} \tilde{C} \lambda_e r_p}{4\pi \gamma \sigma_0^2},$$

is the tune shift expected for the unperturbed initial cloud density [2].

## EXTENSION TO NON-LINEAR TRANSVERSE FORCE

Via a simple tracking code, we extended the analysis to electrons moving in the potential of a transverse Gaussian beam. For the simulations we took the parameters for LHC at injection, listed in Table 1.

Table 1: Parameters used in the simulations for LHC at injection

electron cloud density	$\rho_e$	$6 \times 10^{11} \text{ m}^{-3}$
bunch population	$N_b$	$1.1 \times 10^{11}$
rms bunch length	$\sigma_z$	0.115 m
rms beam size	$\sigma_b$	0.884 mm
nominal tunes	$Q_{x,y}$	64.28, 59.31
electron cloud size	$\sigma_0$	$10 \sigma_b$
electron initial velocity	$\dot{\sigma}_0$	$\omega_e \sigma_0 / 100$

Figure 2 shows the electron density evolution at the centre of the pipe, during the passage of a bunch, computed with the linear force approximation and for the Gaussian beam profile. The simulation with the linear force acting on the electrons is consistent with the analytical prediction (the height of the peaks depends on the initial electron velocity, which was slightly different; there is also a small shift in the position of the peaks, which depends on the initial condition and the slicing in our simulation).

On the other hand, if the electrons move in the potential generated by a transversely Gaussian beam, the peaks almost disappear and after a quarter oscillation the density stays almost constant. In the case of the non-linear force, in fact, as illustrated in Fig. 3, the electrons do not reach the centre of the bunch simultaneously, but their oscillation frequency depends on the initial amplitude. Figure 4 displays snapshots of the radial distribution of the electrons at different times during the bunch passage both for the linear force approximation and for the Gaussian potential. Figure 5 is the same as the right picture in Fig. 2, but on a linear vertical scale. Figure 6 shows the phase space distribution for the same (nonlinear) case, at four different time steps, which are indicated in Fig.5.

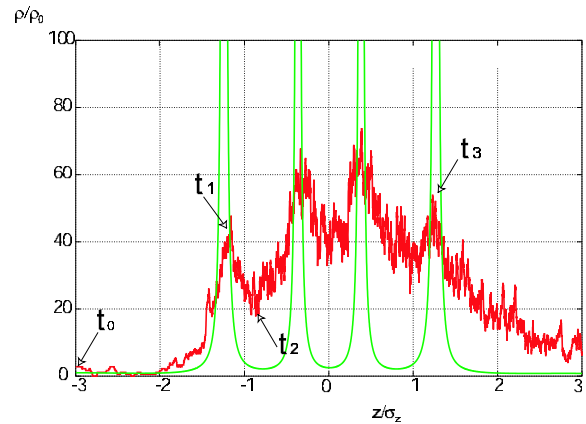


Figure 5: Electron density vs. time at the centre of the pipe, during the passage of a Gaussian bunch (non-linear force).

As can be seen in Figs. 5 and 2, the density enhancement at the center of the bunch is about a factor 50. This allows

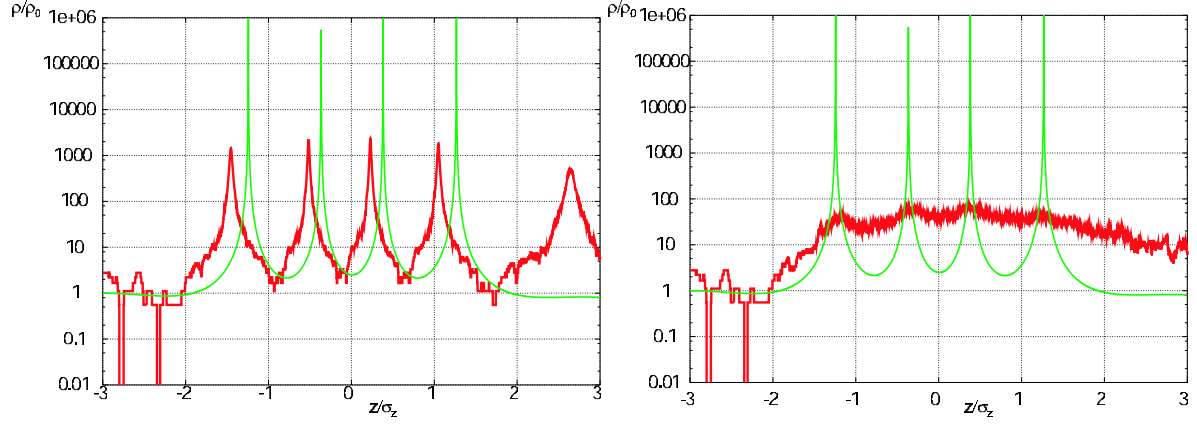


Figure 2: Electron density vs. time at the centre of the pipe, during the passage of a bunch, assuming a linear transverse force (left) and a Gaussian transverse beam profile (right). In red the simulated density evolution and in green the analytical results. A Gaussian bunch profile is assumed in  $z$ .

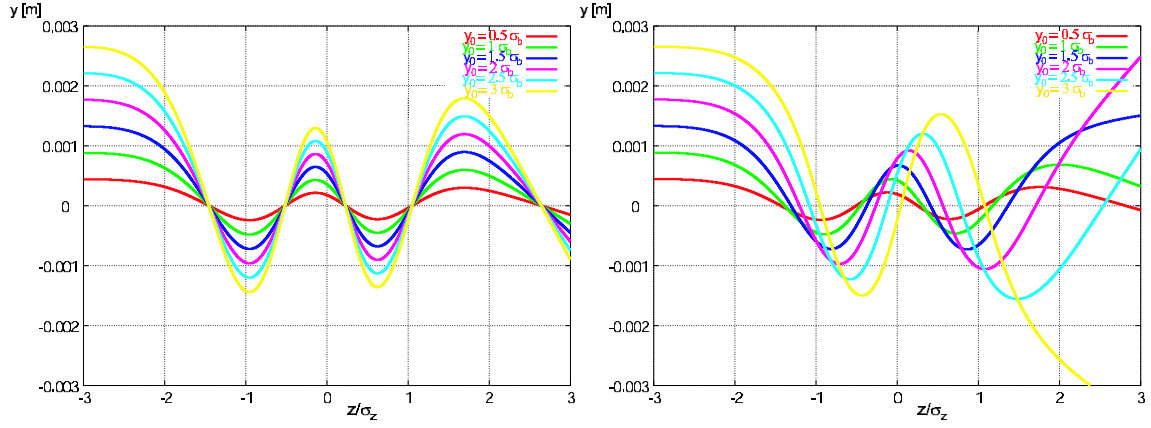


Figure 3: Vertical position versus time for 6 electrons starting at different amplitudes spaced at  $0.5\sigma$ : (left) linear force; (right) Gaussian force. A Gaussian bunch profile is assumed in  $z$ .

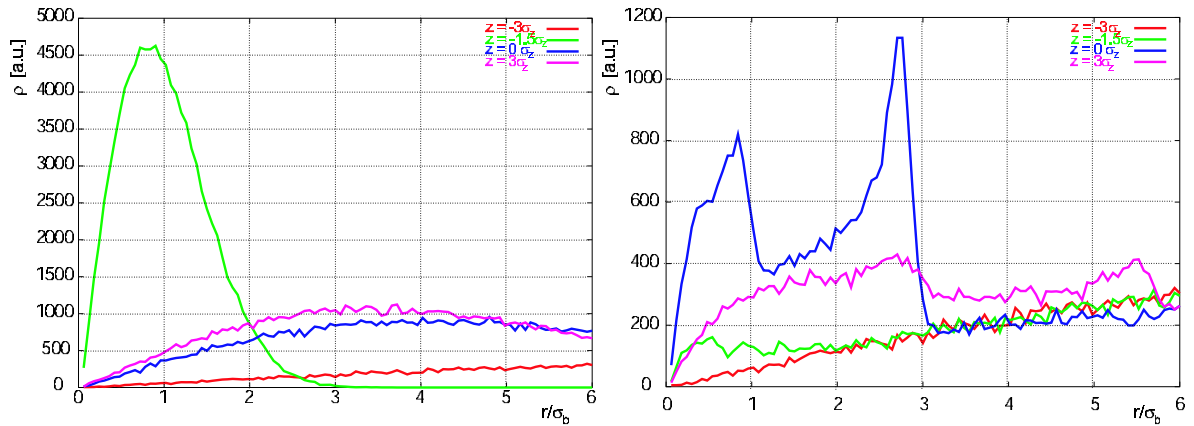


Figure 4: Snap shots of radial distribution ( $\rho \times r$ ) at 4 different times during the bunch passage: (left) linear force approximation, (right) Gaussian transverse profile. A Gaussian bunch profile is assumed in  $z$ .

us to roughly estimate the tune spread via

$$\Delta Q \approx \frac{\tilde{\beta} \tilde{C} r_p}{2\gamma} n_e, \quad (51)$$

where  $n_e$  denotes the enhanced electron density. For the example of the LHC, this gives the value  $\Delta Q \approx 0.13$ , if the initial unperturbed electron cloud density is  $6 \times 10^{11} \text{ m}^{-3}$ . A frequency map analysis [3] from HEADTAIL simulations [4] in a frozen-field approximation gave a tune spread of  $\approx 0.05$  at  $z = +2\sigma_z$ . This tune spread corresponds to a density enhancement of a factor 20 [5] (see Fig. 7), in nearly perfect agreement with Fig. 5.

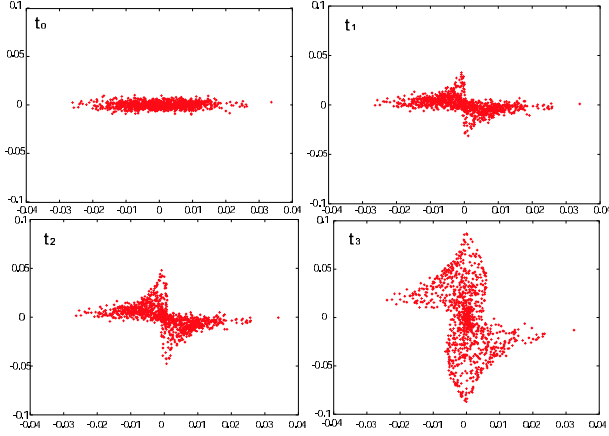


Figure 6: Horizontal phase space at different time steps (which are marked in Fig.5):  $t_0$  is when the bunch enters into the cloud ( $z = -3\sigma_z$ ),  $t_1$  correspond to the first peak,  $t_2$  the first ‘valley’ and  $t_3$  is the last peak.

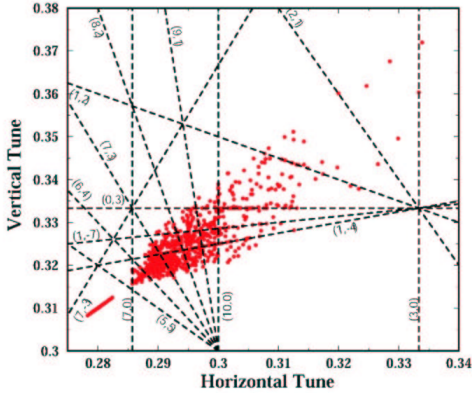


Figure 7: Tune footprint obtained by tracking through a frozen electron potential by a frequency-map analysis [5] (Courtesy Y. Papaphilippou).

## SUMMARY AND OUTLOOK

We presented an analytical approach to compute the incoherent tune shift caused by the electron pinch during the

passage of a bunch through the electron cloud. An expression for the electron density evolution was derived for any longitudinal bunch profile, a linear transverse force, and circular symmetry. The incoherent tune shift as a function of radial and longitudinal position inside the bunch has been computed from the pinched electron distribution.

Via a simple tracking code, we have extended this study to electrons moving in the nonlinear field of a beam with Gaussian transverse profile. In this case, the electrons do not reach the centre of the bunch simultaneously, and after a quarter oscillation the density at the center of the bunch stays roughly constant. It is easy to estimate the tune shift from the value of this stationary density enhancement.

In the near future we plan to continue with analytical approaches to model the electron cloud phenomena, e.g., by generalizing our calculation to the nonlinear force. Our objective is in particular to understand the causes (and even the very existence) of a slow emittance growth on a scale longer than the synchrotron period [6, 7, 8, 9]. We will further compare analytical results with HEADTAIL [4] simulations. We also plan to produce instability diagrams for the electron cloud. Other challenging open questions are the possible excitation of plasma waves by a passing bunch and, related, the effect of longitudinal discontinuities in the electron plasma.

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